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Canonical Proper Time Formulation of Relativistic Particle Dynamics

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A canonical (contact) transformation is performed on the time variable (in extended phase space) to reexpress relativistic dynamics in terms of proper time, leaving the generalized coordinates and canonical momentum as functions of this time variable. The form of the energy functional conjugate to this time variable is seen to be similar to that of a nonrelativistic dynamics at all values of particle momenta. The formulation is explored for single- and multiparticle classical systems. The (form) invariance of the theory is determined by a group which results from a similarity action of the contact group on the Poincaré group. One advantage of this approach is that it overcomes the no-interaction difficulties introduced by standard methods.

1. INTRODUCTION

1.1. Background

In 1864, 1 year before the end of the American Civil War, James Clark Maxwell submitted his theory of electrodynamics (Maxwell, 1865, 1981); 15 years later Albert Einstein was born. In the intervening 41 years between Maxwell and the introduction of the special theory of relativity in 1905, a scientific revolution had taken firm roots. Maxwell's theory had provided answers to almost all major questions in electromagnetism and optics. From the practical point of view, the electric light had been invented and electricity was well on the way toward providing what we now consider normal. On the other hand, the work of Newton (1687, 1966) and his contemporaries was already well understood and it was believed that complete understanding only awaited sufficient mathematics to assist in tidying up

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the theory. Mechanics at this time was in the hands of engineers and mathematicians and much of it was not considered mainstream physics. From this point of view, it comes as no surprise that when problems arose in the interface between mechanics and electromagnetism, the physics community was more than ready to keep the Maxwell theory intact and seek modification of the Newtonian theory.

When Einstein (1950*a*,*b*) and his contemporaries began to study the issues associated with the foundations of electrodynamics, they had a number of options open to them in addressing the fact that the Newtonian theory and Maxwell theory were invariant under different transformation groups:

1. Both theories were incorrect and the proper theory was yet to be found.
2. The Maxwell theory was incorrect and the proper theory would be invariant under the Galilean group.
3. The Maxwell theory was correct and a proper Newtonian theory would be invariant under the Lorentz group.
4. The assumption of an ether for electromagnetic propagation was correct, so that Galilean relativity applied to mechanics, while electromagnetism had a preferred reference frame.

At the time it was unthinkable that the Maxwell theory had any serious flaws. Lorentz had recently shown that all the macroscopic phenomena of electrodynamics and optics could be accounted for based on the analysis of the microscopic behavior of electrons and ions [Lorentz (1903); for the original papers see Lorentz (1892)].

Einstein rejected the fourth possibility and proposed that all physical theories should satisfy the (now well-known) postulates of special relativity:

1. The physical laws of nature and the results of all experiments are independent of the particular inertial frame of the observer (in which the experiment is performed).
2. The speed of light is independent of the motion of the source.

The first postulate abandons the notion of absolute space, while the second abandons absolute time. It is important to note that another postulate is required in order to implement the above two postulates:

3′. The correct implementation of postulates 1 and 2 requires that we represent time as a fourth coordinate, and constrain the relationship between components so as to satisfy the natural invariance induced by the Lorentz group (of electromagnetism); Minkowski space.

This third postulate was made by Minkowski, a well-known mathematician, and was embraced by many. Others (including Einstein) regarded it as a mathematical abstraction lacking physical content. The feeling among many of the leading physicists at that time was that an alternate implementation should be possible which preserved some remnant of an "absolute time" variable while still allowing for the constancy of the speed of light. The works of Ritz (1908*a*,*b*, 1912) and Tolman (1910*a*,*b*) are notable in this direction. The inability to obtain a viable alternative directed by physical considerations forced acceptance of the current implementation.

1.2. Problems

It is clear that something is amiss in that a recurring set of serious problems have continued to block the successful implementation of the first two postulates. Of course the one-particle theory is trivial; however, the two-body theory is still a major problem. It should be noted that there does not exist a relativistic analog of the (Newtonian) reduction of the two-body problem (Rohrlich, 1979; Dirac, 1977; Rosen, 1969). Pryce (1948) showed that there were serious problems with center-of-mass invariance for any attempted implementation. The no-interaction theorem of Currie (Currie *et al.*, 1963) showed that the reasonable assumptions of Hamiltonian formulation, independent (canonical) particle variables, and invariance under the Lorentz group were only compatible with noninteracting particles. This problem has continued to prevent the construction of a satisfactory interacting N-body relativistic theory. Van-Dam and Wigner (1965, 1966) gave up a Hamiltonian formulation and generalized earlier work of Wheeler and Feynman (1945, 1949) in order to bypass the difficulty. Since it is very difficult to conceive of a mechanics without an "energy function," this approach has not been acceptable to all; furthermore, such approaches present other problems when one attempts quantization; see, however, the work of Hoyle and Narlikar (1969) and Davis (1970) (see also Pegg, 1979).

An alternate approach which has current favor is to relax the requirement of canonical variables (Kerner, 1972; Rohrlich, 1979; Longhi and Lusanna, 1986). The problem with this approach is that it leads to many possible theories. Recent work by Longhi and Lusanna (1986) has shown that many of the most actively studied approaches are locally equivalent. This cannot be considered satisfactory until global equivalence is established (since we hope that physical reality is unique). The recent work of Longhi *et al.* (1989) is particularly noteworthy in that they derive relationships between canonical and noncanonical variables by taking a novel many-times approach.

1.3. Purpose

The purpose of this paper is to propose an alternate implementation of the first two postulates without introducing time as a fourth coordinate. Our approach is based on the observation that we may use the invariant proper time variable in place of the observer time variable in the description of the system dynamics. To be sure, the use of this variable is not new; however, we treat the transformation from observer time to system proper time as a canonical (contact) transformation on extended phase space. This approach forces the identification of the canonical Hamiltonian which generates the Lie algebra bracket. This leads to a conceptually (and technically) much simpler implementation of the special theory of relativity.

2. SINGLE-PARTICLE FORMULATION

The dynamics of a classical observable can be conveniently studied using Hamiltonian dynamics. The Poisson bracket is defined as

$$\{A(p, q), B(p, q)\} \equiv \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}$$

The Hamilton equations $\partial H/\partial p = \dot{q}$ and $\partial H/\partial q = -\dot{p}$ then ensure that the time development of an arbitrary classical function $W(q, p, t)$ is given by

$$\frac{dW}{dt}(q, p, t) = \{H, W(q, p, t)\} + \left(\frac{\partial W}{\partial t}\right)_{q,p} \tag{1}$$

Next, define the proper time τ through the relation

$$dt = \frac{H}{mc^2}\,d\tau \tag{2}$$

The time evolution of the function W is given by the chain rule

$$\frac{dW}{d\tau} = \frac{dW}{dt}\frac{dt}{d\tau} = \frac{H}{mc^2}\{H, W\} + \left(\frac{\partial W}{\partial \tau}\right)_{q,p}$$

An energy functional K conjugate to the time τ will be defined to satisfy

$$\begin{aligned} \{K, W\} &= \frac{H}{mc^2}\{H, W\} \\ K\,\big|_{H=mc^2} &= H = mc^2 \end{aligned} \tag{3}$$

Henceforth units will assume $c = 1$. If the mass m remains invariant during the dynamics, the form of the functional K can be directly determined as

$$K = \frac{H^2}{2m} + \frac{m}{2} \tag{4}$$

or, more generally, if m_0 is a well-defined mass point

$$K = m_0 + \int_{m_0}^{H} \frac{dt}{d\tau}\, dH' = m_0 + \int_{m_0}^{H} \frac{H'}{m}\, dH' \tag{5}$$

The evolution of the function W in terms of τ can be expressed now as follows:

$$\frac{dW}{d\tau} = [K, W\} + \frac{\partial W}{\partial \tau} \tag{6}$$

Consider the behavior of a single, noninteracting particle of mass m, with momentum p as measured in some inertial frame. The usual form of the Hamiltonian representing this system is $H = (p^2 + m^2)^{1/2}$. For this example, the conjugate proper energy is given by $K = p^2/2m + m$. Several interesting points should be noted:

a. The functional form of the energy K is the same as that of the nonrelativistic energy of the system, even though the system is fully relativistic.

b. The momentum parameter in the functional form of the energy K is the momentum as measured in the original inertial frame, not the proper frame of the particle (which of course would measure zero momentum). This reemphasizes the form of the transformation as a canonical time transformation, and not a Lorentz transformation.

c. If the particle were to interact with external influences, the proper frame would not be an inertial frame, but the proper time is always defined, and in fact is the only true time relevant to the particle itself.

d. As a particular example of equation (6), the "Hamilton equations" for the q and p variables are given in terms of time τ by

$$\frac{dq}{d\tau} = \frac{\partial K}{\partial p}, \qquad \frac{dp}{d\tau} = -\frac{\partial K}{\partial q} \tag{7}$$

e. The troublesome square root in the Hamiltonian is absent in the form of K.

3. TRANSFORMATION GROUP

As was noted earlier, the proper time is invariant for all inertial observers; however, different observers will use different Hamiltonians to describe the phase flow of the system. In order to relate the phase flows from different inertial observers we note that the proper-time transformations are a subgroup of the full group of transformations on the extended phase space. It should be further noted that this subgroup includes the group of symplectic diffeomorphisms (they do not transform time).

Consider two inertial observers in frames $\mathbf{X}$ and $\mathbf{X}'$ with (extended) phase space coordinates $(\mathbf{p}, \mathbf{q}, t)$ and $(\mathbf{p}', \mathbf{q}', t')$, respectively (for the dynamics of some system). We let $\mathscr{P}$ be the set of Poincaré transformations on space-time reference frames in particular, $(P(X, X'): X \to X')$. We denote by $\mathbb{C}$ the set of canonical proper-time transformations defined on extended phase space. We let the map from $(\mathbf{p}, \mathbf{q}, t) \to (\mathbf{p}, \mathbf{q}, \tau)$ be denoted by $\mathbb{C}(\mathbf{q}, t, \tau)$.

Theorem 1. The proper-time coordinates $\mathbf{X}$ are related to those on $\mathbf{X}'$ by the transformation

$$\mathbf{S}_m[\mathbf{q}', \mathbf{q}, \tau] = \mathbb{C}[\mathbf{q}', t', \tau]\mathscr{P}_m(\mathbf{X}, \mathbf{X}')\mathbb{C}^{-1}[\mathbf{q}, t, \tau] \tag{8}$$

Proof. The proof follows from the obvious commutativity of the following diagram:

$$\begin{tikzcd} \overset{(\mathbf{p}, \mathbf{q}, t)}{\mathbf{X}} \arrow[r, "\mathscr{P}_m(\mathbf{X}, \mathbf{X}')"] \arrow[d] & \overset{(\mathbf{p}', \mathbf{q}', t')}{\mathbf{X}'} \arrow[d] \\ (\mathbf{p}, \mathbf{q}, \tau) \arrow[r, dashed, "\delta_m(\mathbf{q}', \mathbf{q}, \tau)"] & (\mathbf{p}', \mathbf{q}', \tau') \end{tikzcd}$$

It is easy to prove that for each fixed (observed) system, the set of proper-time transformations between inertial observers is a group which relates the dynamics viewed by one (inertial) observer to those of any other.

We use the mass as a subscript in order to fix the observed system. The group of proper-time transformations depends on 14 parameters $(m, \mathbf{p}, \mathbf{q}, \mathbf{p}', \mathbf{q}', \tau)$. It follows that the (free-particle) laws will be the same for all inertial observers and will be (form) invariant under a similarity group action on the Poincaré group.

3. VARIATIONAL APPROACH

In order to obtain the general functional for the contact transformation, we can use the fundamental theorem on the integral invariant of Poincaré–Cartan, $d\omega = \mathbf{p} \cdot d\mathbf{q} - H\,dt$, which states:

Theorem 2 (Arnold, 1978, p; 237). If two curves γ_1 and γ_2 encircle the same tube of phase trajectories of the equations

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}, \qquad \frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} \tag{9}$$

then the integrals of the form $d\omega = \mathbf{p} \cdot d\mathbf{q} - H\,dt$ along γ_1 and γ_2 are equal,

$$\oint_{\gamma_1} (\mathbf{p} \cdot d\mathbf{q} - H\,dt) = \oint_{\gamma_2} (\mathbf{p} \cdot d\mathbf{q} - H\,dt)$$

This theorem can be used to prove:

Theorem 3 (Arnold, 1978, p. 241). Let $(\mathbf{P}, \mathbf{Q}, T)$ be a coordinate system on the extended phase space $(\mathbf{p}, \mathbf{q}, t)$ with $K(\mathbf{P}, \mathbf{Q}, T)$ and $S(\mathbf{P}, \mathbf{Q}, T)$ functions such that $p \cdot d\mathbf{q} - H\,dt = \mathbf{P} \cdot d\mathbf{Q} - K\,dT + dS$. Then the trajectories of the phase flow of equation (9) are represented in the coordinates $(\mathbf{P}, \mathbf{Q}, T)$ by the integral curves of the canonical equations

$$\frac{d\mathbf{P}}{dT} = -\frac{\partial K}{\partial \mathbf{Q}}, \qquad \frac{d\mathbf{Q}}{dT} = \frac{\partial K}{\partial \mathbf{P}}, \qquad \frac{dK}{dT} = \frac{\partial K}{\partial T} \tag{10}$$

For a noninteracting particle $[H = (p^2 + m^2)^{1/2}]$ the corresponding conjugate variables with $dT = d\tau = dt/\gamma$ are

$$\mathbf{q}(t), \qquad \mathbf{p} = \gamma m \frac{d\mathbf{q}}{dt}$$

$$\mathbf{Q}(\tau) = \mathbf{q}(t(\tau)), \qquad \mathbf{P} = m \frac{d\mathbf{Q}}{d\tau}$$

so

$$\mathbf{p}(t(\tau)) = \mathbf{P}(\tau), \qquad \mathbf{q}(t(\tau)) = \mathbf{Q}(\tau), \qquad dt = \gamma\, d\tau$$

The integral invariant satisfies

$$\begin{aligned} \mathbf{p}(t) \cdot d\mathbf{q}(t) - H\,dt &= \mathbf{P}(\tau) \cdot d\mathbf{Q}(\tau) \\ &= \mathbf{P}(\tau) \cdot d\mathbf{Q}(\tau) - \left(\frac{H^2}{2m} + \frac{m}{2}\right) dt \\ &\quad + \left(\frac{m}{2} - \frac{H^2}{2m}\right) d\tau \end{aligned} \tag{11}$$

It follows that $dS = (m/2 - H^2/2m)\,d\tau$; thus the transformation is truly canonical iff dS is a total differential (closed 1-form). Since m is positive and $H^2/2m$ does not change sign as a function of $\tau(t)$, it follows that $\oint_\gamma dS = 0$.

It might be somewhat enlightening to examine alternate functional forms for the conjugate proper functional K. For a system of mass (rest energy) M and three-velocity $\mathbf{v}$ the energy-momentum satisfies $E^2 - \mathbf{P} \cdot \mathbf{P} = M^2$ (*invariant*).

Three methods will be examined for the integration in Equation (5):

a. Hold M fixed, allow the $\mathbf{v}$ Lorentz frame to vary:

$$K_{(1)}(H) = \frac{H^2}{2M} + \frac{M}{2}$$

which is the previously obtained result.

b. Hold the three-momentum $\mathbf{P} = \mathbf{P}_0$ fixed, allow H, M, and $\mathbf{v}$ to vary:

$$K_{(2)}(H) = [H^2 - P_0^2]^{1/2} = M$$

This form represents the proper energy functional as the rest energy of the system. The integration serves to build up the mass and velocity of the system from an initial mass point M_0 at fixed momentum $\mathbf{P}_0$.

c. Hold the $\mathbf{v}$ (Lorentz) frame fixed, allow H, $\mathbf{P}$, and M to vary:

$$K_{(3)}(H) = \frac{H^2}{M}$$

Here, the integration builds up the mass from an initial mass point M_0. This form was derived in Gill (1982).

The advantages or problems associated with each of these forms will be examined briefly later, as regards the system decomposition in multiparticle systems.

Next, consider the application of the above formulations to a system with an internal one-parameter Abelian gauge symmetry group (like electromagnetism). If the energy H and the momentum $\mathbf{P}$ satisfy $H^2 - \mathbf{P} \cdot \mathbf{P} = m^2$, where m is a Lorentz invariant, then we can still write the connection between the inertial and a proper time as follows:

$$dt = \frac{H}{m} d\tau \tag{12}$$

Since the energy is not generally gauge invariant, one expects the ratio of energy to the energy in the *zero-momentum frame* as defined above not to be gauge invariant. Alternatively, one could define the gauge-invariant ratio

$$dt = \frac{H - e\phi}{m^* - e\phi^*} d\tau \tag{13}$$

where the quantities $m^* - e\phi^*$ represent the form of $H - e\phi$ in the *zero-velocity frame*, where as usual the velocity is given by $dq/dt = \{H, q\} = \partial H/\partial p$. The gauge coupling constant is represented by e in the previous expressions, and ϕ represents the potential component of a covariant form of the gauge potential. This frame is not generally equivalent to the zero-momentum frame. For instance, in a formulation in which

$$\dot{q} = \frac{\mathbf{p} - (e/c)\mathbf{A}}{H - e\phi}$$

the momentum vanishes in the frame in which the velocity vanishes only at positions at which the vector potential vanishes. One can see that far from external interactions, the mass m^* is equivalent to m; however, the general connection, both in terms of implicit dependencies as well as Lorentz frame of definition, is generally complicated.

The identification in equation (12) preserves the form of the noninteracting equations, and will be used henceforth. The mass in this equation can be interpreted as the zero-momentum form of the energy, which is held constant during the "boosts" which define the conjugate energy K in equation (5). The process to construct K using the identification in equation (13) is more complicated and will not be further explored in this paper. It should be noted that one could choose a gauge in which the single component $A^0 = \phi$ vanishes in the inertial frame of reference. Of course, this choice will not result in a Lorentz-covariant representation of the gauge field. With this choice, the form of the previous equations is again preserved, and interactions are included. However, since the gauge is not covariant, the form of the term $m^* - e\phi^*$ would involve some interpretations as to how the "boosts" to generate K are performed. We will therefore assume in what follows that the Hamiltonian for interacting systems has been expressed in a way in which the inertial and proper-time connection can be expressed as in equation (2).

5. MULTIPARTICLE FORMULATION

Next, the formulation will be examined in terms of subsystems and clusters of a classical system. The dynamics will be described in terms of the generalized coordinates and momenta $\mathbf{q} = \{q_r\}$ and $\mathbf{p} = \{p_r\}$, and a multiparticle Poisson bracket

$$\{A(\mathbf{p}, \mathbf{q}), B(\mathbf{p}, \mathbf{q})\} = \sum_r \left(\frac{\partial A}{\partial p_r}\frac{\partial B}{\partial q_r} - \frac{\partial A}{\partial q_r}\frac{\partial B}{\partial p_r}\right)$$

with corresponding Hamilton equations $\partial H/\partial p_r = \dot{q}_r$, $\quad \partial H/\partial q_r = -\dot{p}_r$.

The invariant zero-momentum energy of the system will be assumed to satisfy the condition

$$M^2 \equiv \left(\sum_s H_s\right)^2 - (\mathbf{P})^2 \tag{14}$$

for subsystems labeled by energy H_s and momentum $\mathbf{p}_s$. The proper time of the system will be written as

$$dt = \frac{H}{M}\, d\tau \tag{15}$$

where the total energy H has been expressed as the sum of the subsystem energies

$$H(\mathbf{p}, \mathbf{q}) = \sum_r H_r(\mathbf{p}, \mathbf{q}, t) \tag{16}$$

The time evolution of a classical function $W(\mathbf{q}, \mathbf{p}, t)$ can be written

$$\begin{aligned} \frac{dW}{d\tau}(\mathbf{q}, \mathbf{p}, t) &= \{K, W\} + \frac{\partial W}{\partial \tau} \\ &= \frac{H}{M} \sum_r \{H_r, W\} + \frac{\partial W}{\partial \tau} \end{aligned} \tag{17}$$

where the bold coordinates represent the set of all q_s and p_s. It is convenient to examine the relationships between the various proper times of the subsystems. Notice that

$$dt = \frac{H}{M}\, d\tau = \frac{H_r}{m_r}\, d\tau_r \qquad \text{for any } r \tag{18}$$

The various formulations of K previously discussed will be examined under this decomposition.

For formulation (1), the functional $K_{(1)}$ satisfies

$$K = \frac{H^2}{2M} + \frac{M}{2} = \frac{1}{2}\left[\frac{dt}{d\tau} \sum_r H_r + M\right] \tag{19}$$

Identifying $1 = (H_r/m_r)\, d\tau_r/dt$, one immediately obtains

$$K = \sum_r \frac{d\tau_r}{d\tau}\left(\frac{H_r^2}{2m_r} + \frac{m_r}{2}\right) + \frac{1}{2}\left(M - \sum_r \frac{d\tau_r}{d\tau} m_r\right) \tag{20}$$

Next, one can identify the zero-momentum total energy M as follows:

$$M = \sum_r \tilde{H}_r = \sum_r \frac{d\tau}{d\tau_r} m_r \tag{21}$$

Thus

$$\begin{aligned} M - \sum_r \frac{d\tau_r}{d\tau} m_r &= \sum_r \frac{d\tau_r}{d\tau}\left[\left(\frac{d\tau}{d\tau_r}\right)^2 - 1\right] m_r \\ &= \sum_r \frac{d\tau_r}{d\tau}\left[\frac{\tilde{H}_r^2}{m_r^2} - 1\right] m_r \\ &= \sum_r \frac{d\tau_r}{d\tau} \frac{\tilde{p}_r^2}{m_r} \end{aligned} \tag{22}$$

Thus, finally,

$$K = \sum_r \frac{d\tau_r}{d\tau} \left[K_r + \frac{\tilde{p}_r^2}{2m_r} \right] \tag{23}$$

Here $\tilde{p}_r$ represents the momentum of the subsystem r as measured in the total zero-momentum frame, and K_r is the canonical proper energy functional for the subsystem r as calculated using the initial inertial coordinates $\mathbf{p}$ and $\mathbf{q}$.

The formulation $K_{(2)} = (H^2 - P_0^2)^{1/2}$ does not simply decompose, because of the square root, and will therefore not be further explored in this context.

The most direct decomposition occurs for the form $K_{(3)}$. By direct and immediate manipulation one obtains

$$K_{(3)} = \frac{H^2}{M} = \sum_r \frac{d\tau_r}{d\tau} \left(\frac{H_r^2}{m_r} \right) = \sum_r \frac{d\tau_r}{d\tau} K_{(3)r} \tag{24}$$

These various decompositions can be interpreted using the appropriate conjugate function K_r, boosted to the overall proper τ frame, while correcting for any appropriate kinetic energy factors due to the difference in the inertial and proper frames of reference.

6. CONCLUSIONS

A canonical formulation of equations of motion has been presented which demonstrates dynamical evolution in terms of particle proper time, without the introduction of time as a fourth coordinate. The equations obtained are form invariant with regard to inertial observers, and the conjugate energy variable can be chosen to eliminate a square root form in the Hamiltonian. The covariance of the equations obtained involves only the use of observer coordinates for position and momenta of the various particles. The full implications of gauge transformations in this formulation remains to be explored.

The formalism presents a straightforward correspondence limit for a quantum mechanical formulation, since the primary operation needed for the construction of evolution parameters involved only Poisson brackets. When the system is quantized, this should present a form-invariant formalism which avoids some of the problems associated with early formulations of relativistic quantum mechanics. With one of the formulations presented ($K_{(1)}$), the dynamics solved takes on the form of standard nonrelativistic (Schrödinger) dynamics, only expressed in terms of the particle proper time (which is generally noninertial).

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